Topological 2D String Theory:
Higher-genus Amplitudes and W_∞ Identities

Debashis Ghoshal[1]

Mehta Research Institute of Mathematics & Mathematical Physics
10 Kasturba Gandhi Marg, Allahabad 211 002, India

Camillo Imbimbo[2]

Theory Division, CERN, CH-1211 Geneva 23, Switzerland[3]

Sunil Mukhi[4]

Tata Institute of Fundamental Research,
Homi Bhabha Rd, Bombay 400 005, India

We investigate Landau-Ginzburg string theory with the singular superpotential X^{-1} on arbitrary Riemann surfaces. This theory, which is a topological version of the $c = 1$ string at the self-dual radius, is solved using results from intersection theory and from the analysis of matter Landau-Ginzburg systems, and consistency requirements. Higher-genus amplitudes decompose as a sum of contributions from the bulk and the boundary of moduli space. These amplitudes generate the W_∞ algebra.

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[1] E-mail: ghoshal@mri.ernet.in

[2] E-mail: imbimbo@vxcern.cern.ch

[3] On leave from *INFN, Sezione di Genova, Genoa, Italy*

[4] E-mail: mukhi@theory.tifr.res.in

1. Introduction

The most interesting solvable string theory corresponds to a two spacetime dimensional background for the propagation of the bosonic string. Its solution via matrix models has provided useful information about string dynamics to all orders in string perturbation theory[1].

The topological reformulation of this theory has provided an alternate route to understand the matrix model results, at least for the special case of compactification of one dimension at the self-dual radius. The first such reformulation[2], as a topological Kazama-Suzuki type coset model[3], explained the origin of the remarkable fact that the genus-g partition function is proportional to the Bernoulli numbers. In addition, some genus-0 amplitudes could be computed in this approach. This work also elucidated some facts about the $c = 1$ string which had remained mysterious for some time, such as the existence of the so-called KPZ[4] description based on $SL(2, R)$ current algebra, and the apparent similarity between the $c = 1$ string and coset models describing two-dimensional black holes.

Subsequently, a different topological reformulation was proposed, in Ref.[5] (and independently in Ref.[6]) in terms of the infrared fixed-point of a Landau-Ginzburg (LG) type topological theory, with a superpotential X^{-1}. This had the advantage that genus-0 amplitudes could be computed rather easily and shown to agree with matrix model results. (For related work, see Ref.[7].) However, the full solution of the theory, in the very elegant form of W_∞ constraints on the partition function, was still lacking from the topological approach.

In the present work, we use a combination of results from matter LG systems in higher genus and intersection theory on the moduli space of Riemann surfaces, along with consistency requirements, to argue that this approach gives a topological derivation of W_∞. (Other approaches to higher-genus have been discussed recently in Ref.[8].)

In particular, this derivation confirms explicitly what had earlier been argued indirectly: that the topological models really do correspond to matrix models at the special self-dual value of the radius. Since genus-0 correlators are insensitive to the radius of compactification, it is necessary to obtain higher-genus correlators and compare them with matrix models to directly deduce the value of the radius.

More generally, the importance of a topological derivation of higher-genus amplitudes is the following. From matrix models at the self-dual radius, one can not only obtain

W_∞ identities which summarize all the tachyon correlators, but also recast them in a Kontsevich-Penner integral form which is closely related to topological properties of the moduli space of Riemann surfaces[9]. However, it remains quite unclear what is the fundamental feature of string theory that is responsible for the emergence of these topological properties, and this is greatly clarified once the same results are shown to follow from a manifestly topological reformulation of the string theory itself. In view of the fact that a topological symmetry algebra underlies all string backgrounds, this is likely to provide a direction to understand general topological features of string theory itself, independent of the background.

2. Review: Amplitudes on the sphere

Consider the topologically twisted Landau-Ginzburg theory with a single superfield X and superpotential $W = -1/X$. In the unperturbed theory, tachyons T_k are given by powers X^{k-1} for k any integer. Tachyons with $k > 0$, called positive tachyons, behave analogously to the primaries of the conventional polynomial Landau-Ginzburg models. They can be used to perturb the superpotential, and generate the so-called small phase space.

In this phase space, the perturbed superpotential $W(X,t)$ and tachyons $T_k(X,t)$ (where t represents the perturbing parameters t_k) are determined by their flow equations

$$\frac{\partial}{\partial t_{k_i}} T_{k_j}(X,t) = C_{W(X,t)}\left(T_{k_i}(X,t), T_{k_j}(X,t)\right)$$
$$\frac{\partial}{\partial t_{k_i}} W(X,t) = T_{k_i}(X,t), \qquad \text{for } k_i > 0, k_j \in \mathbb{Z} \tag{2.1}$$

Explicit solution of the equations above shows that the tachyons with positive momenta are constants of the flow, $T_k(X,t) = X^{k-1}$, $k \geq 1$, while the potential is a linear function of the couplings:

$$W(X,t) = -X^{-1} + \sum_{k=1}^{\infty} t_k T_k = -X^{-1} + \sum_{k=1}^{\infty} t_k X^{k-1} \tag{2.2}$$

Tachyons with negative momenta T_{-n}, $n > 0$ can be solved for order by order in t, and in analogy with the polynomial LG theories, can be expressed as:

$$T_{-n}(X,t) = \left(\frac{(-W(X,t))^n}{-n}\right)'_{-} \tag{2.3}$$

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where the minus-subscript denotes the terms which are negative powers of X.

The simplest correlator is the genus-0 3-point function $\langle\langle T_{k_1} T_{k_2} T_{k_3}\rangle\rangle_0$ for which the moduli space is a single point. The expression for the correlation function is therefore the same as that in the matter theory. Consider first the case where there is only one negative momentum tachyon, $k_1 = -n, k_2, k_3 \geq 0$ and use the flow equations (2.1) and (2.3) to write

$$\langle\langle T_{-n}(t) T_{k_2} T_{k_3}\rangle\rangle_0 = \oint \frac{1}{W'} \left(\frac{(-W)^n}{-n}\right)'_- \frac{\partial W}{\partial t_{k_2}} \frac{\partial W}{\partial t_{k_3}} \tag{2.4}$$

We can remove the minus-subscript above since the positive terms do not contribute to the integral. Then using the fact that the positive momentum tachyons are constants of the flow, we integrate (2.4) above to write the one-point function of a negative tachyon T_{-n}, $n > 0$ (in the small phase space) as

$$\langle\langle T_{-n}\rangle\rangle_0 = \frac{1}{n(n+1)} \oint (-W)^{n+1} = \oint \frac{1}{W'} \left[\frac{1}{n(n+1)} \left(\frac{(-W)^{n+2}}{-(n+2)}\right)'\right] \tag{2.5}$$

The N-point correlation function of one negative and $N-1$ positive tachyons follows by differentiating this equation. We find

$$\langle\langle T_{-n} T_{k_2} \cdots T_{k_N}\rangle\rangle_0 = \prod_{j=1}^{N-3} (j-n) \oint \frac{1}{W'} \left(\frac{(-W)^{n-N+3}}{-(n-N+3)}\right)' \partial_{t_{k_2}} W \cdots \partial_{t_{k_N}} W \tag{2.6}$$

The especially simple form of correlators with a single negative tachyon will enable us, later on, to provide an interpretation of this expression as the integral of a top form on the $(N-3)$-dimensional moduli space of the N-punctured sphere.

The other three-point functions in the small phase are the ones with two or three negative tachyons:

$$\langle\langle T_{-n_1}(t) T_{-n_2}(t) T_{k_3}\rangle\rangle_0 = \oint \frac{1}{W'} \left(\frac{(-W)^{n_1}}{-n_1}\right)'_- \left(\frac{(-W)^{n_2}}{-n_2}\right)'_- \frac{\partial W}{\partial t_{k_3}}$$

$$\langle\langle T_{-n_1}(t) T_{-n_2}(t) T_{-n_3}\rangle\rangle_0 = \oint \frac{1}{W'} \left(\frac{(-W)^{n_1}}{-n_1}\right)'_- \left(\frac{(-W)^{n_2}}{-n_2}\right)'_- \left(\frac{(-W)^{n_3}}{-n_3}\right)'_- \tag{2.7}$$

With this information, one can write the superpotential in the big phase space, where the negative tachyons T_{-k} are also taken to generate perturbations with parameters $\bar{t}_k$:

$$W(t, \bar{t}) = -X^{-1} + \sum_{k>0} t_k X^{k-1} + \sum_{k>0} \bar{t}_k \left(\frac{(-W(t))^k}{-k}\right)'_-$$

$$+ \frac{1}{2} \sum_{k_1,k_2>0} \bar{t}_{k_1} \bar{t}_{k_2} \left(\frac{\left(\frac{(-W(t))^{k_1}}{-k_1}\right)'_- \left(\frac{(-W(t))^{k_2}}{-k_2}\right)'_-}{W'(t)}\right)'_- + \mathcal{O}(\bar{t}^3) \tag{2.8}$$

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Remarkably, it turns out that to this order, the above expression for $W(t, \bar{t})$ is equivalent to

$$W(t) = -X^{-1} + \sum_{k>0} t_k X^{k-1} - \frac{1}{\mu^2} \sum_{k>0} k \langle\langle T_k \rangle\rangle X^{-k-1} \tag{2.9}$$

where now the $\bar{t}$ dependence is implicit in the last term. To see, this, differentiate both sides in $\bar{t}$ and set $\bar{t} = 0$ to get

$$
\begin{aligned}
\frac{\partial}{\partial \bar{t}_n} W \big|_{\bar{t}=0} &= -\sum_{k>0} k X^{-k-1} \langle\langle T_k T_{-n} \rangle\rangle \\
&= -\sum_{k>0} k X^{-k-1} \oint \frac{(-W(Y))^n}{-n} Y^{k-1} dY \\
&= -\oint \left(\frac{(-W(Y))^n}{-n} \right)_- \frac{1}{(X-Y)^2} dY \\
&= \left(\frac{(-W(X))^n}{-n} \right)'_-
\end{aligned}
\tag{2.10}
$$

as expected. Similarly, differentiating twice gives the last term in Eq.(2.8).

Let us introduce the partition function of the theory in the big phase space,

$$Z(t, \bar{t}) = e^{\mu^2 F(t, \bar{t})}, \tag{2.11}$$

where $F(t, \bar{t})$ is the free energy, and μ the cosmological constant. $F(t, \bar{t})$ has a genus expansion, $F = \sum_{g=0}^{\infty} \frac{1}{\mu^{2g}} F_g$, and in this section we restricted ourselves to the sphere contribution F_0 of order zero in μ.

Since $\langle T_k \rangle = \frac{\partial}{\partial t_k} F$, Eq.(2.9) implies that the perturbed superpotential, when multiplying $Z(t, \bar{t})$ can be substituted by the operator-valued superpotential $\hat{W}$, at lowest order in $1/mu^2$:

$$\hat{W}(t) = -X^{-1} + \sum_{k>0} t_k X^{k-1} - \frac{1}{\mu^2} \sum_{k>0} k \frac{\partial}{\partial t_k} X^{-k-1}. \tag{2.12}$$

In this form, we say that the superpotential is "quantized". The expression for $\hat{W}(X)$ is precisely the mode expansion of a free spin-1 current in a conformal invariant field theory on the complex plane whose complex coordinate is the Landau-Ginzburg superfield X. (This fact looks very natural if, as suggested in Ref.[5], the LG superpotential is identified with the quantized string field.)

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We can succintly rewrite the Eqs. (2.5) and (2.7) for the correlation functions on the sphere, in the small phase space, with at most three negative tachyons, in terms of the "quantized" superpotential as follows:

$$\frac{1}{\mu^2} \frac{\partial}{\partial \bar{t}_n} Z = \frac{1}{n(n+1)} \oint \, : (-\hat{W})^{n+1} : Z. \tag{2.13}$$

This is the matrix model result in Ref. [9] restricted to genus zero. We have just seen that the flow equations (2.1) in the small phase space (which originate from the contact term prescription in the unperturbed theory) are sufficient to demonstrate Eq.(2.13) up to third order in $\bar{t}$ in the free energy $F_0(t,\bar{t})$. In Ref.[6], it has been shown that one can prescribe an infinite set of multi-contact terms, such that Eq.(2.9) is true to all orders in $\bar{t}$. This in turn implies that, in genus-0, Eq.(2.13) is true with $\hat{W}$ given by Eq.(2.12).

The contact terms required for this theory (both the "simple-contact" appearing in Eq.(2.1) and the multi-contacts mentioned above) have not so far been actually derived from the Landau-Ginzburg Lagrangian, but must be taken as an ansatz which gives a consistent solution. (This is essentially true also of the contact terms appearing in the polynomial LG theories describing $c < 1$ string backgrounds, which were used implicitly in Ref.[10] and explicitly in Ref.[11].) In the present case, since correct genus-0 amplitudes incorporating all contact terms can be summarized in the quantization of the superpotential, it is convenient to assume that quantization of the superpotential is a basic physical property of this theory. A more detailed understanding of the compactified moduli spaces relevant to noncritical strings would be required to provide a rigorous justification for this ansatz.

In the following sections we will, however, show that this ansatz satisfies a very stringent consistency check which is derived by considering the theory at higher genus. In fact Eq.(2.13) is not consistent as it stands at higher genus, since the operators acting on the partition function on the R.H.S.,

$$\mathcal{W}^{(n)} = \frac{1}{n(n+1)} \oint \, : (-\hat{W})^{n+1} :, \tag{2.14}$$

do not satisfy the integrability condition. Their commutators up to terms of order $1/\mu^4$ are

$$\left[\mathcal{W}^{(n)}, \mathcal{W}^{(m)} \right] = \frac{1}{24\mu^4} (n-1)(m-1)(m-n) \oint \, : (-\hat{W})^{n+m-5} (-\hat{W}')^3 : + \mathcal{O}\left(\frac{1}{\mu^6}\right). \tag{2.15}$$

This implies that the higher-genus generalization of Eq.(2.13)

$$\frac{1}{\mu^2}\frac{\partial}{\partial \bar{t}_n}Z = \mathcal{W}^{(n)}(\mu)Z, \tag{2.16}$$

must involve some deformation $\mathcal{W}^{(n)}(\mu)$ of the genus-0 operator $\mathcal{W}^{(n)}$, admitting an expansion of the form

$$\mathcal{W}^{(n)}(\mu) = \mathcal{W}^{(n)} + \frac{1}{\mu^2}\mathcal{W}_1^{(n)} + \frac{1}{\mu^4}\mathcal{W}_2^{(n)} + \cdots \tag{2.17}$$

The higher-genus contributions $\frac{1}{\mu^{2g}}\mathcal{W}_g^{(n)}$ should be such that the deformed operators $\mathcal{W}^{(n)}(\mu)$ commute. However, it is known that, assuming that the $\mathcal{W}^{(n)}(\mu)$ are polynomials in the operator valued superpotential $\hat{W}(X)$ and its derivatives, the integrability condition determines *uniquely* the higher-genus contributions $\mathcal{W}_g^{(n)}$, order by order in $\frac{1}{\mu^2}$. For example it easy to verify that the genus-1 contribution $\mathcal{W}_1^{(n)}$ which cancels the commutator in Eq.(2.15) up to order $\frac{1}{\mu^4}$ is

$$\mathcal{W}_1^{(n)} = \frac{1-n}{24}\oint :(-\hat{W})^{n-2}\hat{W}'' : . \tag{2.18}$$

The higher-genus Ward identities (2.16) have been derived from matrix-models in Ref. [9]. Explicit expressions for the operatorial coefficients $\mathcal{W}_g^{(n)}$ for any genus g are derived in the Appendix.

In the following sections we will derive the one-point functions of negative tachyons in the small phase space starting from the analysis of the LG topological theory on higher-genus world-sheet surfaces. The higher-genus contributions to the negative tachyon one-point functions we compute agree precisely with the ones determined by the Ward identities (2.16) involving the deformed operators in Eq.(2.17). We regard this as a higher-genus, non-trivial verification of the superpotential "quantization" ansatz. Moreover, our results strongly suggest that the LG topological theory with the singular superpotential X^{-1} does in fact define a consistent world-sheet quantum field theory.

3. Picture-changing of Negative Tachyons

We have already pointed out that expressions for N-point functions, such as Eq.(2.6) above, should have an interpretation in a topological theory as the integral of a top form over the relevant moduli space. The specially simple form of Eq.(2.6), where there is a

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single negative tachyon, allows us to find the desired interpretation. This equation is of the form of a matter Landau-Ginzburg correlator[10][12] where the positive tachyons appear in their usual form, but the negative tachyon, in an N-point function, appears as

$$T_{-n} \sim \prod_{j=1}^{N-3}(j-n)\left(\frac{(-W)^{n-N+3}}{-(n-N+3)}\right)' \tag{3.1}$$

(Recall that the minus-subscript was dropped since it does not contribute in the presence of only positive tachyons.) In the absence of perturbations, this field carries an effective "momentum" of $-n + N - 3$ units. For the special case of three-point functions, where the moduli space is trivial, the above expression reduces to the standard one for negative tachyons. Thus we can interpret Eq.(2.6) as saying that, in genus 0, the negative tachyon appears in many different "pictures", with the contribution for a particular correlator coming entirely from the relevant picture in which momentum is correctly conserved.

The reason why this picture-changing hypothesis makes sense is the following: the topological conservation law says that it is not momentum alone which is conserved, but the difference between momentum and form dimension, where a gravitational secondary σ_n carries form dimension n. Thus in order to view the negative tachyon as carrying a different momentum in different pictures, it is necessary to compensate by also assigning it a different form dimension in each picture, so that the contribution to the conservation law is the same independent of the picture.

The precise statement of this is that the negative tachyon is really a sum of gravitational descendants above the fields T_{-n} that we have been working with above, and reduces to T_{-n} only when the moduli space is trivial. Thus we write

$$\mathcal{T}_{-n} = \sum_{i=0}^{n} \prod_{j=1}^{i}(j-n)\sigma_i(T_{-n+i}) \tag{3.2}$$

where the T_{-n} are given in Eq.(2.3) (without the minus-subscript, as we have already remarked). Note that the numerical factor on the RHS of this expression can independently be found[5] by an appropriate continuation of the argument, due to Losev[11], that relates matter-sector fields to gravitational secondaries. This lends further support to the above identification. Henceforth we interpret this expression as the physical negative tachyon.

This notion of picture-changing neatly solves the puzzle we raised above, that the N-point function should be a form of dimension $N - 3$ integrated over moduli space. The correlation function of one negative and $N - 1$ positive tachyons should now be calculated

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using $\mathcal{T}_{-n}$ above to represent the negative tachyon. Of the various terms in $\mathcal{T}_{-n}$, the only one which will contribute is the one in which on the one hand, the matter sector conserves momentum, while on the other hand the gravitational sector has the form dimension appropriate to make it a top form on the moduli space. It is evident that both requirements are satisfied simultaneously, and by just one of the terms in Eq.(3.2) above. Since the positive tachyons are all gravitational primaries, the relevant sphere correlation function in the gravitational sector is $\langle \sigma_n \sigma_0 \ldots \sigma_0 \rangle$ which is 1 when the moduli space has dimension n and 0 otherwise. Hence the answer reduces to a pure matter-LG calculation and we manifestly recover Eq.(2.6).

The above hypothesis should be thought of as the precise version of a rather general statement made in Refs.[5] and [6] that the negative tachyon behaves as a gravitational secondary. In fact, it is the sum of many gravitational secondaries, of form dimensions varying from 0 to n, of which the first (primary) term contributes for sphere 3-point functions, but the other terms contribute on nontrivial moduli spaces.

As it stands, our proposed picture-changing expression makes sense only for correlators involving a single negative tachyon. When two or more negative tachyons are present, multi-point contact terms will be present and this simple picture will not hold. Fortunately, the fact that these multi-contact terms are summarised in the quantization of the superpotential means that it is enough to do everything for one-point functions of the negative tachyon, in the small phase space where $\bar{t} = 0$. This will lead to the small-phase-space version of the $W_{1+\infty}$ constraints of Ref.[9], from which one can argue that the big phase space result follows upon quantization of the superpotential.

This is not equivalent to suggesting that the equations of the subsequent section can be extended directly to the big phase space; they can be so extended only after rewriting them as operator constraints on the partition function. It would be useful to show directly that multi-negative-tachyon correlators can be correctly derived in the LG approach, but this seems rather difficult and we leave it as an open problem.

4. Higher-genus Correlators

In higher-genus, the moduli space is always nontrivial. We will use the expression Eq.(3.2) and some simple facts about matter LG systems to obtain the correlation functions. The basic relation we will need is that if $\mathcal{O}$ is any operator in a matter LG theory, then genus-g and genus-0 correlators are related as follows[12],[13]:

$$\langle \mathcal{O} \rangle_g = \langle (W'')^g \mathcal{O} \rangle_0 \tag{4.1}$$

where the second derivative of the superpotential, W'', can be thought of as the "handle operator".

Our point of view will be that correlators of the LG theory coupled to gravity can be constructed using the above expression to represent the handles, and appropriately picture-changed operators to represent the insertion of the tachyon field. At this stage the computation factorizes into a matter-like contribution in the form of a contour-integral (containing the matter part of the picture-changed tachyon) and the correlation functions of the gravitational operators σ_n, which are computed from pure topological gravity[14] (equivalent to intersection theory on moduli space).

For example, in genus 1 with one puncture, the moduli space is 1-dimensional, and we have

$$
\begin{aligned}
\langle\langle \mathcal{T}_{-n}\rangle\rangle_{g=1}(\bar{t}=0) &= \langle \sigma_1 \rangle_{g=1} \oint \frac{W''}{W'}(1-n)\left(\frac{(-W)^{n-1}}{-(n-1)}\right)' \\
&= \frac{1-n}{4!}\oint W''(-W)^{n-2}
\end{aligned}
\tag{4.2}
$$

Here we have used the result from Ref.[14] that $\langle \sigma_1 \rangle_{g=1} = \frac{1}{4!}$. The above expression is precisely the torus one-point function of a negative tachyon as obtained from matrix models, at $\bar{t}=0$!

Before going on to look at general genus, it is useful to write down the known answers from matrix models for ease of comparison. The one-point function of a negative tachyon at $\bar{t}=0$ is given in Ref.[9] as a function of the cosmological constant μ. The genus-g contribution is obtained by expanding in powers of $1/\mu$ and keeping the term of order μ^{-2g} relative to the tree level term. This is worked out in the Appendix. The result is a combinatorial formula involving a sum over partitions of g, of the form $1^{\alpha_1}2^{\alpha_2}\cdots g^{\alpha_g}$ with $\sum_{l=1}^{g}l\alpha_l = g$. Let $p = \sum_{l=1}^{g}\alpha_l$ be the total number of elements in the partition, then

$$
\langle\langle \mathcal{T}_{-n}\rangle\rangle_g(\bar{t}=0) = \sum_{\substack{\alpha_1,\ldots,\alpha_g \\ \sum \alpha_l = p,\ \sum l\alpha_l = g}} \langle\langle \mathcal{T}_{-n}\rangle\rangle_g^{(\alpha_1,\ldots,\alpha_g)}
$$

$$
\langle\langle \mathcal{T}_{-n}\rangle\rangle_g^{(\alpha_1,\ldots,\alpha_g)} = \frac{1}{2^{2g}}\frac{\prod_{j=1}^{2g-2+p}(j-n)}{\prod_{l=1}^{g}\alpha_l!\,((2l+1)!)^{\alpha_l}}
\tag{4.3}
$$

$$
\oint \frac{\prod_{l=1}^{g}(\partial^{2l}W)^{\alpha_l}}{W'}\left(\frac{(-W)^{n-2g+2-p}}{-(n-2g+2-p)}\right)'
$$

(Here, ∂ represents $\partial/\partial X$.)

In particular, the contribution from the specific partition with $\alpha_1 = g, \alpha_2, \ldots = 0$, for which $p = g$, is

$$\langle\langle \mathcal{T}_{-n} \rangle\rangle_g^{(g,0,\ldots,0)} = \frac{1}{2^{2g}} \frac{\prod_{j=1}^{3g-2}(j-n)}{g!(3!)^g} \oint \frac{(W'')^g}{W'} \left(\frac{(-W)^{n-3g+2}}{-(n-3g+2)} \right)' \tag{4.4}$$

but in general there are several other terms, except in genus 1 where the above term gives the whole answer. From this we see in particular that the genus 1 expression Eq.(4.2) calculated from LG theory agrees with the matrix model result.

Recalling Eq.(3.2), one sees that in every term of Eq.(4.3) above, a picture-changed tachyon appears, but each time in a different picture. Indeed, for a partition into p elements, the tachyon appears in the $2g-2+p$ picture, so it must be thought of as a $2g-2+p$ form on moduli space. For the special partition which contributes to Eq.(4.4), we have $p = g$ and hence the tachyon appears here in the $3g - 2$ picture, which corresponds to the dimension of the moduli space $\mathcal{M}_{g,1}$. We conclude that this is the unique term in the matrix model amplitude which can be thought of as an integral of a top form over the moduli space of genus g with one puncture. The other terms must correspond to boundaries of $\overline{\mathcal{M}}_{g,1}$.

Returning now to the Landau-Ginzburg theory, we generalize to arbitrary genus the calculation leading to Eq.(4.2), to find

$$\langle\langle \mathcal{T}_{-n} \rangle\rangle_g(\bar{t} = 0) = \langle \sigma_{3g-2} \rangle_g \oint \frac{(W'')^g}{W'} \prod_{j=1}^{3g-2} (j-n) \left(\frac{(-W)^{n-3g+2}}{-(n-3g+2)} \right)' \tag{4.5}$$

From pure topological gravity it is known[14] that

$$\langle \sigma_{3g-2} \rangle_g = \frac{1}{g!(4!)^g} \tag{4.6}$$

Inserting this in Eq.(4.5), one finds that this LG result is precisely equal to Eq.(4.4) in every genus, including all factors!

Thus the LG theory reproduces the "bulk" term in every genus. To get the other terms, one has to analyse the various boundaries of $\overline{\mathcal{M}}_{g,1}$ and see whether the extra terms in Eq.(4.3) can be interpreted as arising from boundary contributions, or in other words as contact terms between handles. We turn to this in the following section.

5. Boundary of Moduli Space and Handle Contact Terms

Consider the case of genus-2, which is the first one for which the bulk contribution to the tachyon amplitude is not the whole answer. From Eq.(4.3) it is easy to work out that the genus-2 amplitude is

$$
\begin{aligned}
\langle\langle \mathcal{T}_{-n}\rangle\rangle_{g=2} =& \frac{1}{(4!)^2 2}(1-n)(2-n)(3-n)(4-n) \oint \frac{(W'')^2}{W'}\left(\frac{(-W)^{n-4}}{-(n-4)}\right)' \\
&+\frac{1}{4^2 5!}(1-n)(2-n)(3-n) \oint \frac{W''''}{W'}\left(\frac{(-W)^{n-3}}{-(n-3)}\right)'
\end{aligned}
\tag{5.1}
$$

The second term above is our first example of a boundary term. Clearly it comes from a boundary, of complex dimension 3, of the dimension-4 moduli space $\overline{\mathcal{M}}_{2,1}$.

This leads us to postulate contact terms between the handle operators W'' of the Landau-Ginzburg theory, analogous to the contact terms between tachyon operators[11] that were crucial in Refs.[5],[6] to define the theory consistently. We have mentioned at the end of section 2 that the handle contact terms are essential for the $\mathcal{W}^{(n)}$ generators to commute, hence they are indeed required by consistency.

To start with, we propose that the contact term between a pair of handles is

$$
C(W'', W'') = \alpha \, W''''
\tag{5.2}
$$

where α is a numerical constant. We call this a 'simple contact', since we will see that there are also multiple contacts. Geometrically we imagine that a contact between a pair of handles arises when two handles coincide, so that a nontrivial homology cycle is pinched. The surface acquires two extra punctures and reduces its genus by one, hence the complex dimension of the moduli space goes down by one as desired.

In genus 2, the contact term above will therefore give a contribution

$$
\langle\langle \mathcal{T}_{-n}\rangle\rangle_{g=2}^{contact} = \langle \sigma_3 \sigma_0 \sigma_0 \rangle_{g=1} \oint \frac{C(W'', W'')}{W'}(1-n)(2-n)(3-n)\left(\frac{(-W)^{n-3}}{-(n-3)}\right)'
\tag{5.3}
$$

where the tachyon has appeared in the "3"-picture. The two σ_0 operators are associated to the two extra punctures on the collapsed surface, which has genus 1. Using the recursion relations of pure topological gravity (in particular, the string equation) we find

$$
\langle \sigma_3 \sigma_0 \sigma_0 \rangle_{g=1} = \langle \sigma_1 \rangle_{g=1} = \frac{1}{4!}
\tag{5.4}
$$

Inserting this and the contact term in Eq.(5.3), we get

$$\langle\langle\mathcal{T}_{-n}\rangle\rangle_{g=2}^{contact} = \frac{\alpha}{4!}(1-n)(2-n)(3-n)\oint\frac{W''''}{W'}\left(\frac{(-W)^{n-3}}{-(n-3)}\right)' \tag{5.5}$$

Comparing with Eq.(5.1) determines the arbitrary constant appearing in the contact term to be

$$\alpha = \frac{1}{5.2^4} \tag{5.6}$$

As a check, we now reproduce the boundary terms in arbitrary genus associated to the contact of a single pair of handles. This is equivalent to pinching of a single nontrivial homology cycle, connecting any two handles. There are $g-1$ such cycles, and the associated partition of g in Eq.(4.3) is clearly $\alpha_1 = g - 2, \alpha_2 = 1, \alpha_3 = \cdots = 0$. Thus we find the contribution

$$\langle\sigma_{3g-3}\sigma_0\sigma_0\rangle_{g-1}\prod_{j=1}^{3g-3}(j-n)\oint(g-1)\frac{C(W'',W'')(W'')^{g-2}}{W'}\left(\frac{(-W)^{n-3g+3}}{-(n-3g+3)}\right)' \tag{5.7}$$

Using the string equation, which gives

$$\langle\sigma_{3g-3}\sigma_0\sigma_0\rangle_{g-1} = \langle\sigma_{3g-5}\rangle_{g-1} = \frac{1}{(4!)^{g-1}(g-1)!} \tag{5.8}$$

the contribution becomes

$$\frac{1}{2^{2g}}\frac{\prod_{j=1}^{3g-3}(j-n)}{5!(3!)^{g-2}(g-2)!}\oint\frac{W''''(W'')^{g-2}}{W'}\left(\frac{(-W)^{n-3g+3}}{-(n-3g+3)}\right)' \tag{5.9}$$

which is seen to agree with $\langle\langle\mathcal{T}_{-n}\rangle\rangle_g^{(g-2,1,0,...,0)}$ of Eq.(4.3) above.

Thus we have found that a simple handle-contact term reproduces a class of contributions to the tachyon amplitude in every genus, corresponding to the pinching of a single non-trivial cycle. Suppose now we allow the pinching of two such nontrivial cycles, then there are two ways that this can happen: either the two pinched cycles link two completely disjoint pairs of handles, or they link two adjacent pairs of handles (so one handle is common). In the first case, the simple contact term suffices to describe this effect, and leads to answers containing two factors of W'''', with some fixed coefficients which can be computed. In the second case, we have a fusing of three handles, and will have to postulate a new contact term to describe this.

Let us look at the first case in more detail. In this case, we are concerned with a boundary of complex codimension 2. To do the counting, we must first assume that we

have pinched one of the relevant cycles, and within this boundary (of codimension 1) we have to count the number of ways in which a second cycle can be pinched. Requiring that the second cycle be disjoint from the first, it can link any two out of $g-2$ possible handles, so the combinatorial factor is $\frac{(g-2)(g-3)}{2}$. (In the second case, to which we return later, the region is again a boundary of codimension 2, and a similar counting gives a factor $g-2$.) With this factor, it is straightforward to show that the term $\langle\langle \mathcal{T}_{-n}\rangle\rangle_g^{(g-4,2,0,\ldots,0)}$ of Eq.(4.3) above is reproduced by using a pair of simple contacts. This can be further generalized to all terms corresponding to partitions of the type $(g-2m,m,0,\ldots,0)$, describing the pinching of m disjoint cycles.

To reproduce partitions for which $\alpha_l \neq 0$ for $l \geq 3$, we need multi-contact terms which represent the fusing of 3 or more handles. For example, the multi-contact betwen three handles produces a term with 6 derivatives of the superpotential. Following the same procedure as for the simple contact, one finds

$$C(W'',W'',W'') = \frac{4!}{7!\,2^6} W'''''' \tag{5.10}$$

The interpretation of this term is that it comes from the boundary of moduli space where two adjacent non-trivial cycles are pinched. In such terms, the tachyon will appear in the picture relevant to the dimension of this boundary, which is $3g-4$, which is of complex codimension 2 just as for the pinching of non-adjacent cycles. This agrees with the analysis of the appropriate terms in Eq.(4.3) containing the sixth derivative of the superpotential.

More generally, the multi-contact between n handles is

$$C(W'',W'',\ldots,W'') \ (n \text{ times}) = \frac{4!}{(2n+1)!\,2^{2n}} \partial^{2n} W \tag{5.11}$$

It is easy to see that this correctly reproduces, for example, the term in genus-g where all the g handles collide, degenerating to a genus-1 surface. In Eq.(4.3), pick the term in genus-g with $\alpha_g = 1$ and all the other $\alpha_i = 0$. The 4! in Eq.(5.11) above is cancelled by the contribution from the gravitational sector:

$$\langle \sigma_{2g-1}(\sigma_0)^{2g-2}\rangle_{g=1} = \frac{1}{4!} \tag{5.12}$$

and it is clear that the other factors in the mtulicontact term are precisely the desired ones. Note that for $n = 1$ the numerical coefficient on the RHS of Eq.(5.11) reduces to unity, as one would expect since in this case there is no contact.

To conclude, the picture-changing operation that we have described, along with the formula Eq.(4.1) for matter Landau-Ginzburg correlators, and the handle-contacts discussed above, reproduce completely the formula Eq.(4.3) for the negative-tachyon 1-point functions, at $\bar{t} = 0$, in every genus.

This means that with the assumptions and computations described above, the formula of Ref.[9] restricted to $\bar{t} = 0$ (Eq.(4.3) above, which is shown in the Appendix to be equivalent to Eq.(8.1)) has been obtained in our topological LG framework. With the quantized superpotential, Eq.(8.1) should be viewed as an operator acting on the partition function of Eq.(2.11) above, in which case it gives all correlators for arbitrary configurations of positive and negative tachyons. This completes the chain of arguments to the effect that the $c = 1$ matrix model results follow from the topological Landau-Ginzburg formulation, modulo the remarks at the end of Section 3.

6. Partition Function

Having obtained the tachyon correlators in every genus from Landau-Ginzburg considerations, we may ask whether this also enables us to find the partition function in each genus. We now show that this is indeed the case.

Although Eq.(4.3) is strictly true for negative tachyons, namely $n \geq 1$, one can check if its limit as $n \to 0$ makes sense. In the original form of Eq.(8.1) of the Appendix (which came from matrix models) this limit appears rather singular. However, in the equivalent form of Eq.(4.3) which was eventually derived from Landau-Ginzburg theory, it is quite easy to take the limit. Restricting to the unperturbed superpotential $W = -1/X$, one finds

$$\langle\langle \mathcal{T}_0 \rangle\rangle_g = \frac{1}{2^{2g}} \sum_{\substack{\alpha_1,\ldots,\alpha_g \\ \sum \alpha_l = p, \sum l\alpha_l = g}} (-1)^p \frac{(2g - 2 + p)!}{\prod_{l=1}^{g} \alpha_l!(2l+1)^{\alpha_l}} \tag{6.1}$$

Since T_0 is the cosmological operator[5], this is equal to $\partial/\partial\mu(Z(\mu))$ expanded in powers of $1/\mu$, where

$$Z(\mu) = \sum_{g=0}^{\infty} Z_g \mu^{2-2g} \tag{6.2}$$

Thus the RHS of Eq.(6.1), which we denote by A_g, equals $(2 - 2g)Z_g$. This gives us an explicit way to compute Z_g.

We now show that A_g is equal to $-B_{2g}/2g$ where B_{2g} are the Bernoulli numbers. Let us write

$$A_g = \sum_{p=1}^{g} A_{g,p}(2g - 2 + p)!,\tag{6.3}$$

which defines the $A_{g,p}$. The corresponding generating function

$$A(z, \lambda) \equiv \sum_{g=0}^{\infty} \sum_{p=1}^{g} A_{g,p} z^p \lambda^{2g}\tag{6.4}$$

becomes

$$
\begin{aligned}
A(z, \lambda) &= \sum_{g,p} (-z)^p \left(\frac{\lambda}{2}\right)^{2g} \sum_{\substack{\alpha_1,\ldots,\alpha_g \\ \sum \alpha_l = p, \sum l\alpha_l = g}} \frac{1}{\prod_{l=1}^{g} \alpha_l!(2l+1)^{\alpha_l}} \\
&= \prod_{l=1}^{\infty} \sum_{\alpha_l=0}^{\infty} \frac{1}{\alpha_l!} \left(-\frac{z}{2l+1} \left(\frac{\lambda}{2}\right)^l\right)^{\alpha_l} \\
&= \exp\left(-z \sum_{l=1}^{\infty} \frac{1}{2l+1} \left(\frac{\lambda}{2}\right)^l\right) \\
&= e^z \left(\frac{1 - \lambda/2}{1 + \lambda/2}\right)^{z/\lambda}.
\end{aligned}\tag{6.5}
$$

Now the idea is to use the Borel resummation trick to obtain the generating function $A(\lambda)$ for the A_g, starting from the known $A(z; \lambda)$:

$$
\begin{aligned}
A(\lambda) &\equiv \sum_{g=0}^{\infty} \lambda^{2g} A_g \\
&= \int_0^{\infty} dt\, e^{-t} \sum_{g=0}^{\infty} \sum_{p=1}^{g} t^{2g-2+p} \lambda^{2g} A_{g,p} \\
&= \int_0^{\infty} \frac{dt}{t^2} e^{-t} A(t; \lambda t) \\
&= \int_0^{\infty} \frac{dt}{t^2} \left(\frac{1 - \lambda t/2}{1 + \lambda t/2}\right)^{1/\lambda},
\end{aligned}\tag{6.6}
$$

Performing the substitution

$$e^{-x} = \frac{1 - \lambda t/2}{1 + \lambda t/2},\tag{6.7}$$

one obtains :

$$A(\lambda) = \frac{\lambda}{4} \int dx\, e^{-x/\lambda} \frac{1}{\sinh^2(x/2)}.\tag{6.8}$$

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Making the identification $\lambda = 1/i\mu$, one verifies that the expansion in inverse powers of μ of this expression is precisely $-B_{2g}/2g$ (see Eqs. (5.36),(5.37) of Ref.[9]).

Thus we have proved that $A_g = -B_{2g}/2g$, so that

$$Z_g = \frac{B_{2g}}{2g(2g-2)} \tag{6.9}$$

which is precisely the partition function originally obtained from matrix models compactified at the self-dual radius[15].

7. Discussion and Conclusions

We have considered the topological Landau-Ginsburg version of 2d string theory at higher genus. The main result of our analysis is that the higher-genus tachyonic correlation functions known from matrix models admit a topological decomposition as a sum of contributions from the interior and from the boundary of moduli space. We also showed that the bulk contribution is easily computed, at all genus, by combining results from topological matter field theory and topological gravity. The contributions from the boundaries of moduli space have been shown to be interpretable as contact terms between collapsing handles. We also extended our topological interpretation of 2d string correlators to the partition function, which at each genus is given by the virtual Euler characteristic of moduli space.

Our work brings the $c = 1$ string into the same topological framework as the $c < 1$ strings. Similarly to the $c < 1$ models, the contribution to the correlation functions coming from the interior of the moduli space is easily determined from topological field theory. As for the contributions from the boundary of the moduli space, they are determined from consistency rather than from strictly field theoretical methods. It would be interesting if these latter contributions could be explicitly re-derived by extending the analysis of Ref.[12] to degenerated Riemann surfaces.

Two features of the topological resolution of the 2d string that we propose appear to be novel. First, it turns out that some of the physical operators of the theory – the negative tachyons – should be picture-changed, not only near the boundary of moduli spaces as in topological gravity[10], but in the bulk part of the correlators as well. Second, the kind of degenerations of Riemann surfaces which give rise to the relevant boundary terms – multi-contacts between collapsing handles – seem to correspond to a compactification of the moduli space which is not the same as the one appearing in topological gravity. It would be interesting to understand both these features from a genuinely field theoretical point of view.

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8. Appendix: Genus Expansion of W_∞ Constraints

In this appendix, we will show that if the generating function of Ref.[9] at $\bar{t} = 0$:

$$\langle\langle T_{-n}\rangle\rangle(\bar{t} = 0) = \oint \frac{1}{n(n+1)} e^{-i\mu\phi(X)} \left(\frac{\partial}{i\mu}\right)^{n+1} e^{i\mu\phi(X)} \tag{8.1}$$

is expanded in inverse powers of μ as follows:

$$\langle\langle T_{-n}\rangle\rangle = \sum_{g=0}^{\infty} (i\mu)^{-2g} \langle\langle T_{-n}\rangle\rangle_g \tag{8.2}$$

then $\langle\langle T_{-n}\rangle\rangle_g(\bar{t} = 0)$ is given by Eq.(4.3) above, where $W(X) = -\partial\phi(X)$ has the mode expansion in Eq.(2.2).

In order to prove Eq.(4.3) we will establish the following, more general result. The operators $\mathcal{W}^{(n)}(\mu)$ appearing in the Ward identities (2.16) derived from matrix-models in Ref.[9] are

$$\mathcal{W}^{(n)}(\mu) = \frac{1}{n(n+1)} : e^{-i\mu\hat{\phi}(X)} \left(\frac{\partial}{i\mu}\right)^{n+1} e^{i\mu\hat{\phi}(X)} : \tag{8.3}$$

where $\hat{W}(X) = -\partial\hat{\phi}(X)$. We will show that the operators $\mathcal{W}_g^{(n)}$ defined by the expansion (2.17) are given by a formula analogous to Eq.(4.3):

$$\begin{aligned}
\mathcal{W}_g^{(n)} = &\frac{1}{2^{2g}} \frac{\prod_{j=1}^{2g-2+p}(j-n)}{\prod_{l=1}^{g} \alpha_l! \left((2l+1)!\right)^{\alpha_l}} \\
&\oint : \frac{\prod_{l=1}^{g}(\partial^{2l}\hat{W})^{\alpha_l}}{\hat{W}'} \left(\frac{(-\hat{W})^{n-2g+2-p}}{-(n-2g+2-p)}\right)' :
\end{aligned} \tag{8.4}$$

When considering the Ward identities (2.16) at $\bar{t} = 0$, one can simply substitute the quantized superpotential $\hat{W}(X)$ with the "classical" superpotential $W(X)$ of Eq.(2.2) in the operatorial expression for $\mathcal{W}_g^{(n)}$. This is because the derivative terms in Eq.(2.12) pull down positive-momentum tachyon correlators from the partition function, but these all vanish at $\bar{t} = 0$. Thus, Eq.(8.4) implies, in particular, the validity of Eq.(4.3).

Introduce the generating operator-valued field $\mathcal{W}(z; \mu)$ defined by

$$\mathcal{W}(z; \mu) \equiv \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n-1)!} \mathcal{W}^{(n)}(\mu) \tag{8.5}$$

Subsituting Eq.(8.3) in the definition above, and using the Taylor expansion formula, we can write it as

$$\mathcal{W}(z; \mu) = \oint : e^{i\mu\hat{\phi}(X+z/i\mu) - i\mu\hat{\phi}(X)} : \tag{8.6}$$

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Since $\mathcal{W}(z; \mu)$ is an even function of μ, it is convenient to make this manifest by shifting $X \to X - z/2i\mu$:

$$\mathcal{W}(z; \mu) = \oint \, : e^{i\mu(\hat{\phi}(X + z/2i\mu) - \hat{\phi}(X - z/2i\mu))} \, : \, \equiv \oint \, : \exp\left(i\mu S(X; z/i\mu)\right) : \, , \qquad (8.7)$$

The "action" $S(X; z/i\mu)$ can be expanded as

$$S(x; z/i\mu) = \sum_{l=0}^{\infty} \frac{2}{(2l+1)!} \left(\frac{z}{2i\mu}\right)^{2l+1} \partial^{2l+1} \hat{\phi}(X). \qquad (8.8)$$

which gives the generating field in the form:

$$\mathcal{W}(z; \mu) = \oint \, : e^{z\partial\hat{\phi}(X)} \exp\left[\sum_{l=1}^{\infty} \frac{z}{(2l+1)!2^{2l}} \left(\frac{z}{i\mu}\right)^{2l} \partial^{2l+1} \hat{\phi}(X)\right] : \qquad (8.9)$$

We now show that Eq.(8.4) leads to the same generating field, thereby proving its equivalence to Eq.(8.3). Inserting this in Eq.(8.2) and Eq.(8.5), and performing first the sum over n, we get

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{(n-1)!} \prod_{j=1}^{2g-2+p} (j-n) : (-\hat{W})^{n-2g+1-p} : = \left(\frac{\partial}{\partial \hat{W}}\right)^{2g+p} \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} : (-\hat{W})^{n+1} :$$

$$= (-z)^{2g+p} : e^{-z\hat{W}} : \qquad (8.10)$$

Using the above in Eq.(8.5), we find

$$\mathcal{W}(z; \mu) = \oint \, : e^{-z\hat{W}} \sum_{g=0}^{\infty} \left(\frac{z}{2i\mu}\right)^{2g} \sum_{\substack{\alpha_1, \ldots, \alpha_g \\ \sum l\alpha_l = g}} \prod_{l=1}^{g} \frac{\left(-z \, \partial^{2l+1}\hat{W}\right)^{\alpha_l}}{\alpha_l! \left((2l+1)!\right)^{\alpha_l}} : \qquad (8.11)$$

One can now use the simple combinatorial identity

$$\exp\left(\sum_{l=1}^{\infty} a_l \lambda^l\right) = \sum_{g=0}^{\infty} \lambda^g \sum_{\substack{\alpha_1, \ldots, \alpha_g \\ \sum l\alpha_l = g}} \prod_{l=1}^{g} \frac{(a_l)^{\alpha_l}}{\alpha_l!} \qquad (8.12)$$

in Eq.(8.11) to show that it is equivalent to the expression (8.9) above, with the identification $\hat{W}(X) = -\partial\hat{\phi}(X)$. Thus we have shown that Eq.(4.3) represents the genus-expansion of the matrix model results.

Acknowledgements

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